Topic 10. Options

- *Derivatives with non-linear payoffs*

Option – a right of a buyer to buy/sell an underlying asset at a specified date (exercise, expiration date) and at a specified price (exercise, strike price). The seller has an obligation.

- Put, call
- European, American
- Underlying assets: stocks, bonds, indices, currencies, commodities, interest rates, futures
- Prices:

➢ Exercise Price (Strike Price): The price in the option contract at which you have the right to buy or sell the underlying asset

➢ Option Premium: The price paid for the option

➢ Intrinsic Value: Value of an option based on immediate exercise. For call option it is the maximum of zero and the difference between the strike price and the stock price.

➢ Time Premium: Value of option above the intrinsic value

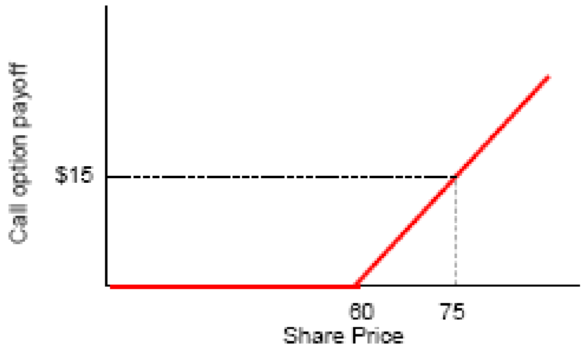
Also: Embedded options (convertible bonds, callable bonds), Exotic options, Swaptions

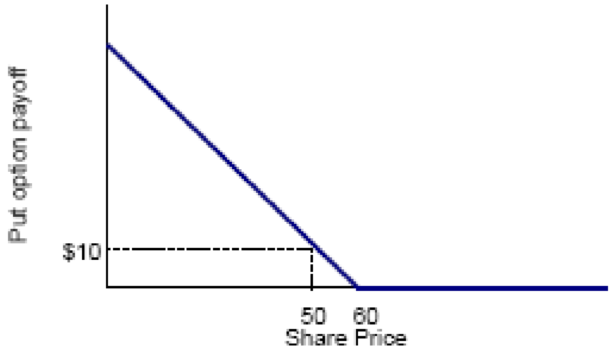
Option payoff

(Assume strike of $60)

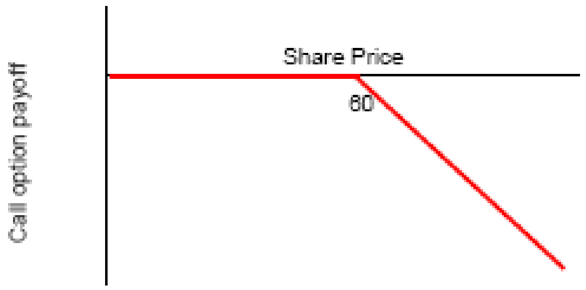
Call option **Put option**

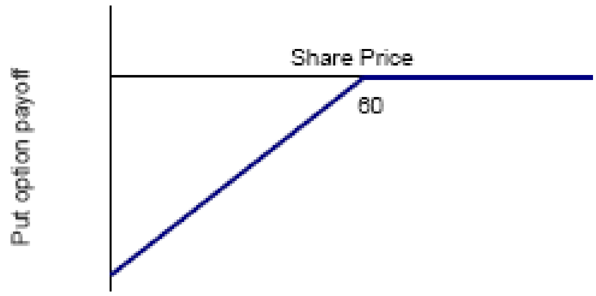
Payoff to buyer:





Payoff to seller:

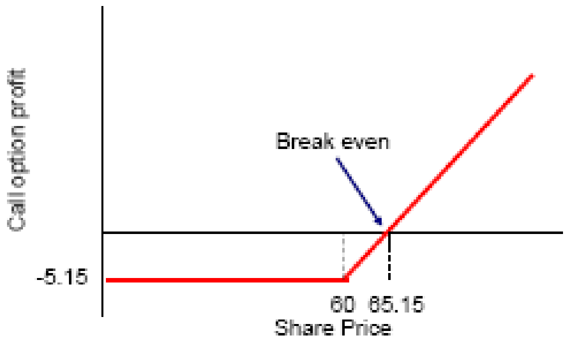




Option profit

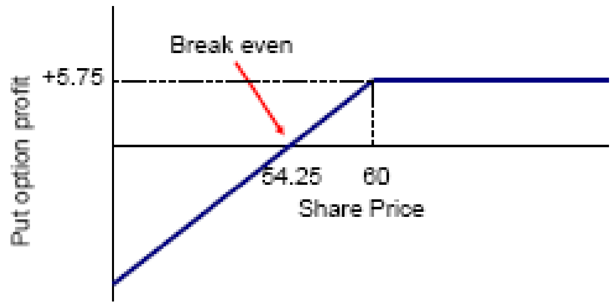
Call buyer profit

Assume premium of $5.15



Put seller profit

Assume premium of $5.75

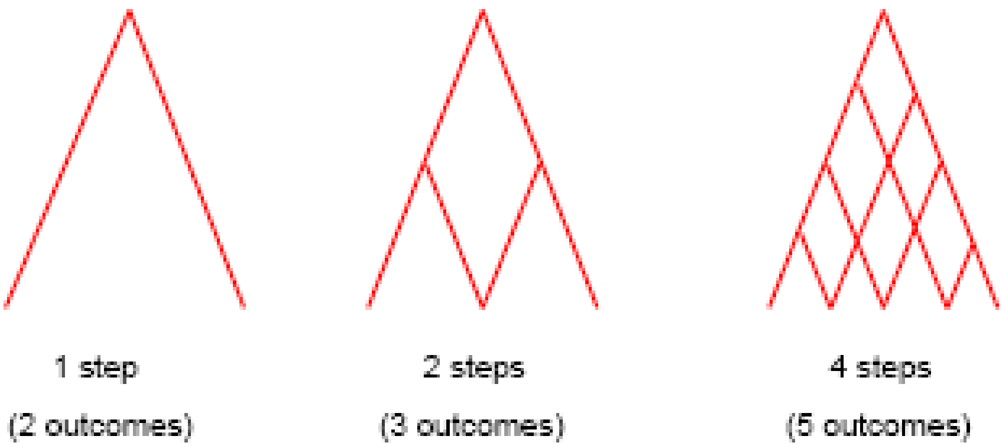


Option pricing

- to find the (fair) option premium
  - ⇨ Method One: Replicating strategy: Find a portfolio of the underlying security plus risk-free asset with the same payoffs and therefore same value
  - ⇨ Method Two: Risk-neutral Probabilities: Create risk-adjusted probabilities instead of true probabilities, and apply risk-free PV with these artificial probabilities

Binomial model

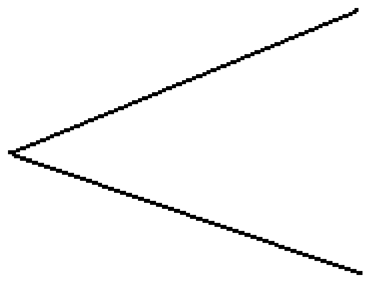
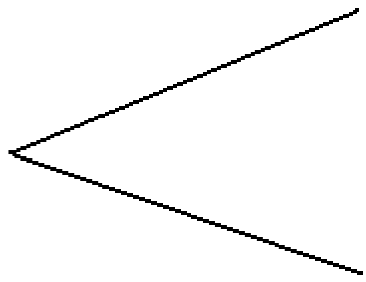
- Every period the stock price goes either up or down with some probabilities



- The more steps we assume, the closer we are to reality.

1-Step Binomial Model

The value of the underlying asset today is S_0. The value of the underlying asset in 1 period will be either S_H or S_L. Consider a call option on the underlying asset with strike X ($S_L<X<S_H$). What is its premium?

Stock payoff	**Call option payoff**
	

Can you replicate the option payoff by holding the underlying asset and borrowing money at r_f?

Replicating strategy:

- Buy(sell) δ shares and borrow (lend) β at r_f

Replicating portfolio payoff:

If stock goes up: $\delta * S_H + \beta * (1+r_f) = S_H - X$

If stock goes down: $\delta * S_L + \beta * (1+r_f) = 0$

Solving the system for δ and β, we find the replicating portfolio: $\delta = \frac{S_H - X}{S_H - S_L}$, $\beta = \frac{S_L(S_H - X)}{(1+r_f)(S_L - S_H)}$.

The value of replicating pfl today equals $\delta * S_0 + \beta$

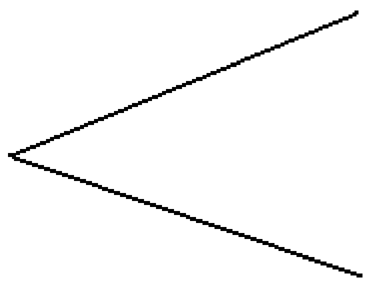
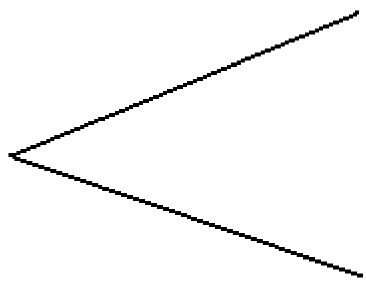
By no arbitrage, option premium MUST be the same:

$$\boxed{C = \delta * S_0 + \beta}$$

1-Step Binomial Model

<u>Example:</u>

- ⇨ Consider a call option on a stock, valued at £50, with an exercise price also equal to £50 (the option is at the money). The annual risk free rate is 1%. Assume that over the year the stock can either rise to £60 or fall to £30
- ⇨ If the stock falls to £30, the call option expires worthless. If it rises to £60, the option will be worth £60-£50=£10

Stock payoff	Option payoff
	

Replicating portfolio payoff:

If stock goes up: $\delta*60+\beta*(1+0.1)=10$

If stock goes down: $\delta*30+\beta*(1+0.1)=0$

Solving the system for δ and β, we find the replicating portfolio: $\delta=1/3$, $\beta=-9.9$.

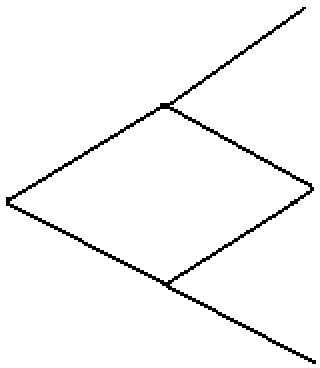
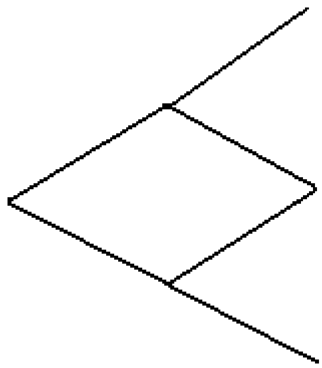
The value of replicating pfl today = $\delta*50+\beta=6.77$

Hence, option premium is also \$6.77.

Find the value of a put option on this stock with the same exercise price.

2-Step Binomial Model

The value of the underlying asset today is S_0. Every period, the value of the underlying asset will either go up u times or go down d times. Consider a European call option on the underlying asset with strike X (assume $ddS_0<X<duS_0$). What is its premium?

Stock payoff	**Option payoff**
T= 0 1 2 	T= 0 1 2 

We go backwards:

Step 1: Find the value V_u

$$\begin{cases} \delta_1 uuS_0 + \beta_1(1+r_f) = uuS_0 - X \\ \delta_1 udS_0 + \beta_1(1+r_f) = udS_0 - X \end{cases} \Rightarrow \delta_1 \text{ and } \beta_1 \Rightarrow V_u=\delta_1 uS_0+\beta_1$$

Step 2: Find the value V_d

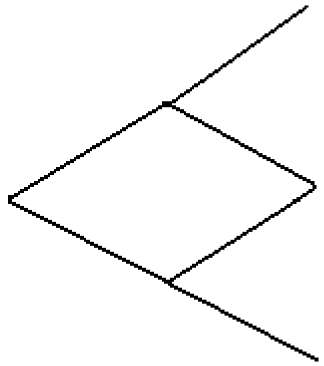
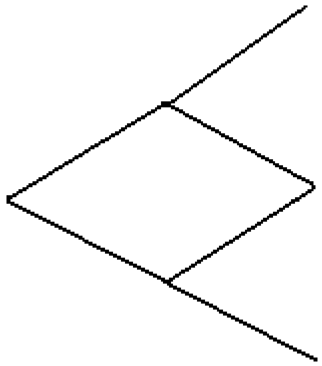
$$\begin{cases} \delta_2 duS_0 + \beta_2(1+r_f) = duS_0 - X \\ \delta_2 ddS_0 + \beta_2(1+r_f) = 0 \end{cases} \Rightarrow \delta_2 \text{ and } \beta_2 \Rightarrow V_d=\delta_2 dS_0+\beta_2$$

Step 3: Find the value C

$$\begin{cases} \delta_3 uS_0 + \beta_3(1+r_f) = V_u \\ \delta_3 dS_0 + \beta_3(1+r_f) = V_d \end{cases} \Rightarrow \delta_3 \text{ and } \beta_3 \Rightarrow C=\delta_3 S_0+\beta_3$$

Pricing American Options

Two-step binomial model. The value of the underlying asset today is S_0. Every period, the value of the underlying asset will either go up u times or go down d times. Consider an <u>American call option</u> on the underlying asset with strike X (assume $dS_0<X<uS_0$ and $ddS_0<X<duS_0$). What is its premium?

Stock payoff	**Option payoff**
T= 0 1 2 	T= 0 1 2 

The same methodology as before (find V_u and V_d), but at T=1 you need to compare the value of the option if exercised <u>now</u> with the value if not exercised now (V_u or V_d). Sometimes it is worth exercising early. This possibility affects the premium.

Can an American option be cheaper than a similar European option?

<u>Risk-neutral pricing</u>

Imagine that investors do not care about risk. Then "no arbitrage" implies that expected return equals r_f:

$$q_u * r_u + (1-q_u) * r_d = r_f$$

q_u – <u>risk-neutral prob.</u> that stock price goes up

$1-q_u=q_d$ - <u>risk-neutral prob.</u> that stock price goes down

q_u and q_d are not ACTUAL probabilities!

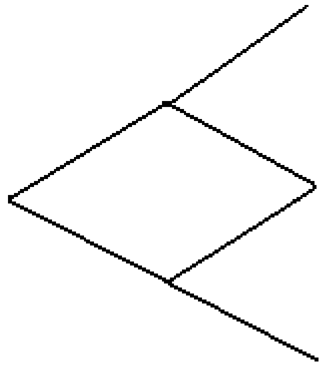
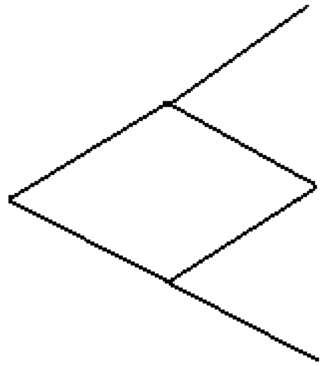
The price of an asset is the PV of expected CF in risk-neutral world discounted at r_f:

$$P = \frac{q_u CF_u + q_d CF_d}{1+r_f} = \frac{E_q[CF]}{1+r_f}$$

Compare this with the standard pricing formula:

$$P = \frac{E_p[CF]}{1+r_{required}}$$

<u>Example:</u> 2-step binomial model, a European call:

Stock payoff	**Option payoff**
T= 0 1 2 	T= 0 1 2 

$$q_u * u + (1-q_u) * d = r_f + 1 \;=>\; \begin{cases} q_u = \dfrac{r_f + 1 - d}{u - d} \\ q_d = -\dfrac{r_f + 1 - u}{u - d} \end{cases}$$

⇨ q_u and q_d are the same in both steps

$$V_u = \frac{q_u(uuS_0 - X) + q_d(udS_0 - X)}{1 + r_f}$$

$$V_d = \frac{q_u(udS_0 - X) + q_d 0}{1 + r_f}$$

$$C = \frac{q_u V_u + q_d V_d}{1 + r_f}$$

Black-Scholes formula (1974)

- derived in continuous time, can be approximated by a binomial model with many periods

$$C = N(d_1)S_0 - N(d_2)Xe^{-r_f t}$$

where N(d) – cumulative normal distribution function (NORMSDIST() f-n in Excel)

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r_f + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{t}$$

<u>Components:</u>

C – call option premium

S_0 – initial spot price of the underlying asset

X – strike price

r_f – risk-free rate

t – time to maturity of option (as % of a year)

σ – standard deviation of returns

To find the value of a similar put use PUT-CALL PARITY

<u>Put-Call Parity</u>

- gives a relationship between prices of a call and a put with similar characteristics

Consider <u>2 strategies</u>:

1) Long position in the underlying asset and a put option
2) Long position in a call option and lending Xe^{-rT}

	Cost	**Pay-off at maturity**
Strategy 1	S_0+P	$S_T+\max[X-S_T,0] = \max[X,S_T]$
Strategy 2	$C+Xe^{-rT}$	$\max[0,S_T-X]+X = \max[X,S_T]$

No-arbitrage condition implies:

$$\mathbf{S_0+P=C+Xe^{-rT}}$$

In discrete time: $\mathbf{S_0+P=C+X/(1+r)^T}$